Distributions on symmetric cones I: Riesz distribution

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Abstract

The Riesz distribution for real normed division algebras is derived in this work. Then two versions of these distributions are proposed and some of their properties are studied.

1 Introduction

In the context of the Statistics and related fields, the analysis on symmetric cones has been developed in depth through the last seven decades. As usual, the studies were focused on the real case, later the complex case appeared and in the recent years, the quaternionic and octonionic numbers were included in the analysis. This analysis has been notorious in the development of certain statistical areas as random matrix distributions, but also in some mathematical topics including special polynomials matrix arguments as zonal, Jack, Davis' invariants, Laguerre, Hermite, Hayakawa; and the associated hypergeometric type series. These mathematical tools have been implemented for example in the characterisation of non-central distributions, statistical theory of shape, etc. Among many authors, we can highlight the works of Herz (1955), James (1961), James (1964), Constantine (1963), Davis (1980), Muirhead (1982), Díaz-García (2011), and the references therein. Of course, the interest in the analysis of symmetric cones does not belong exclusively to statistics; the harmonic analysis, for example, is an special case of the real symmetric cone and this cone play a fundamental role in number theory. Another particular case that has been studied in detail, specially from the point of view of the wave equation, is the Lorentz cone, Faraut and Korányi (1994) and the references therein.

Although during the 60's and 70's were obtained important results in the general analysis of the symmetric cones, the past 20 years have reached a substantial development. Essentially, these advances have been archived through two approaches based on the theory of Jordan algebra and the real normed division algebras. A basic source of the general theory of symmetric cones under Jordan algebras can be found in Faraut and Korányi (1994); and specifically, some works in the context of distribution theory in symmetric cones based on Jordan algebras are provided in Massam (1994), Casalis and Letac (1996), Hassairi and Lajmi (2001), and Hassairi *et al.* (2005), and the references therein. In the field of spherical functions (Jack polynomials including James' zonal polynomials) we can mention the work of Sawyer (1997). Parallel results on distribution theory based on real normed division algebras have been also developed in statistics and random matrix theory, see Forrester

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(2009), Díaz-García and Gutiérrez-Jáimez (2011), Díaz-García and Gutiérrez-Jáimez (2012a), among others. A detailed study of functions with matrix arguments on symmetric cones, such as Jack and Davis' invariants polynomials and hypergeometric types series, among others, appear in Gross and Richards (1987), Díaz-García and Gutiérrez-Jáimez (2012b) and Díaz-García and Gutiérrez-Jáimez (2009), for real normed division algebras.

Under the approach based on the theory of Jordan algebras, a family of distributions on symmetric cones, termed the Riesz distributions, was first introduced by Hassairi and Lajmi (2001) under the name of Riesz natural exponential family (Riesz NEF); it was based on a special case of the so-called Riesz measure from Faraut and Korányi (1994, p.137), going back to Riesz (1949). This Riesz distribution generalises the matrix multivariate gamma and Wishart distributions, containing them as particular cases.

This article introduces the Riesz distribution for real normed division algebras. Section 2 revises some definitions and notations on real normed division algebras, also, two definitions of the generalised gamma funcion on symmetric cones are given; in addition, several Jacobians with respect to Lebesgue measure for real normed division algebras are proposed. Finally, two Riesz distributions are obtained and under both definitions, the corresponding characteristic function and the eigenvalue distribution are derived in section 3.

2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Gross and Richards (1987). For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes, a **vector space** is always a finite-dimensional module over the field of real numbers. An **algebra** $\mathfrak{F}$ is a vector space that is equipped with a bilinear map $m : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$ termed *multiplication* and a nonzero element $1 \in \mathfrak{F}$ termed the *unit* such that $m(1, a) = m(a, 1) = a$. As usual, we abbreviate $m(a, b) = ab$ as ab. We do not assume $\mathfrak{F}$ associative. Given an algebra, we freely think of real numbers as elements of this algebra via the map $\omega \mapsto \omega 1$.

An algebra $\mathfrak{F}$ is a **division algebra** if given $a, b \in \mathfrak{F}$ with $ab = 0$, then either $a = 0$ or $b = 0$. Equivalently, $\mathfrak{F}$ is a division algebra if the operation of left and right multiplications by any nonzero element is invertible. A **normed division algebra** is an algebra $\mathfrak{F}$ that is also a normed vector space with $||ab|| = ||a||||b||$. This implies that $\mathfrak{F}$ is a division algebra and that $||1|| = 1$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see Baez (2002). We take into account that $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of $1, 2, 4$ or 8, which is denoted by β, see Baez (2002, Theorems 1, 2 and 3). In other branches of mathematics, the parameters $\alpha = 2/\beta$ and $t = \beta/4$ are used, see Sawyer (1997) and Kabe (1984), respectively.

Let $\mathcal{L}^{\beta}_{m,n}$ be the set of all $m \times n$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{F}^{m\times n}$ be the set of all $m \times n$ matrices over $\mathfrak{F}$. The dimension of $\mathfrak{F}^{m\times n}$ over $\Re$ is βmn. Let $\mathbf{A} \in \mathfrak{F}^{m\times n}$, then $\mathbf{A}^* = \bar{\mathbf{A}}^T$ denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

We denote by $\mathfrak{S}^{\beta}_{m}$ the real vector space of all $\mathbf{S} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. In addition, let $\mathfrak{P}^{\beta}_{m}$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{F}^{m\times m}$. Thus, $\mathfrak{P}^{\beta}_{m}$ consist of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, with $\mathbf{X} \in \mathfrak{L}^{\beta}_{m,n}$; then $\mathfrak{P}^{\beta}_{m}$ is an open subset of $\mathfrak{S}^{\beta}_{m}$.

Let $\mathfrak{D}^{\beta}_{m}$ be the *diagonal subgroup* of $\mathcal{L}^{\beta}_{m,m}$ consisting of all $\mathbf{D} \in \mathfrak{F}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$.

Table 1: Notation

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}_{m,n}^{\beta}$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^{\beta}(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}_{m}^{\beta}$

For any matrix $\mathbf{X} \in \mathfrak{F}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{F}^{m\times n}, \mathfrak{S}_m^{\beta}$, $\mathfrak{D}_m^{\beta}$ or $\mathcal{V}_{m,n}^{\beta}$, see Díaz-García and Gutiérrez-Jáimez (2011).

If $\mathbf{X} \in \mathfrak{F}^{m\times n}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{F}^{m\times n}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{m}\bigwedge_{j=1}^{n} dx_{ij} \quad \text{ where } \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^{\beta}$ (or $\mathbf{S} \in \mathfrak{T}_U^{\beta}(m)$ is a upper triangular matrix) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_m^{\beta}$ or in $\mathfrak{T}_U^{\beta}(m)$) denotes the exterior product of the exterior product of the $m(m-1)\beta/2+m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii}\bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_m^{\beta}$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^{\beta}$ then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}_m^{\beta}$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n}\bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{m}\bigwedge_{j=i+1}^{n} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1^*|\mathbf{H}_2^*)^* = (\mathbf{h}_1,\dots,\mathbf{h}_m|\mathbf{h}_{m+1},\dots,\mathbf{h}_n)^* \in \mathfrak{U}^{\beta}(n)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_2$ matrix. When $n = 1$; $\mathcal{V}_{m,1}^{\beta}$ defines the unit sphere in $\mathfrak{F}^{m}$. This is, of course, an $(m-1)\beta$- dimensional surface in $\mathfrak{F}^{m}$. When $n = m$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}d\mathbf{H}^*)$ is termed the *Haar measure* on $\mathfrak{U}^{\beta}(m)$.

The multivariate *Gamma function* for the space $\mathfrak{S}_m^{\beta}$ denotes as $\Gamma_m^{\beta}[a]$, is defined by

$$\begin{aligned}\Gamma_m^{\beta}[a] &= \int_{\mathbf{A}\in\mathfrak{P}_m^{\beta}} \operatorname{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1}(d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4}\prod_{i=1}^{m}\Gamma[a-(i-1)\beta/2],\end{aligned}$$

where $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$, $|\cdot|$ denotes the determinant and $\operatorname{Re}(a) > (m-1)\beta/2$, see Gross and Richards (1987). This can be obtained as a particular case of the *generalised*

gamma function of weight κ for the space $\mathfrak{S}_m^\beta$ with $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, taking $\kappa = (0, 0, \dots, 0)$ and which for $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$ is defined by, see Gross and Richards (1987),

$$\begin{aligned} \Gamma_m^\beta[a, \kappa] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A})(d\mathbf{A}) \qquad (1) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2] \\ &= [a]_\kappa^\beta \Gamma_m^\beta[a], \qquad (2) \end{aligned}$$

where for $\mathbf{A} \in \mathfrak{S}_m^\beta$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}} \qquad (3)$$

with $\mathbf{A}_p = (a_{rs})$, $r, s = 1, 2, \dots, p$, $p = 1, 2, \dots, m$ is termed the *highest weight vector*, see Gross and Richards (1987). In other branches of mathematics the *highest weight vector* $q_\kappa(\mathbf{A})$ is also termed the *generalised power* of $\mathbf{A}$ and is denoted as $\Delta_\kappa(\mathbf{A})$, see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Some additional properties of $q_\kappa(\mathbf{A})$, which are immediate consequences of the definition of $q_\kappa(\mathbf{A})$ and the following property 1, are:

1. if $\lambda_1, \dots, \lambda_m$, are the eigenvalues of $\mathbf{A}$, then

$$q_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_i}. \qquad (4)$$

2.

$$q_\kappa(\mathbf{A}^{-1}) = q_\kappa^{-1}(\mathbf{A}) = q_{-\kappa}(\mathbf{A}), \qquad (5)$$

3. if $\kappa = (p, \dots, p)$, then

$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p, \qquad (6)$$

   in particular if $p = 0$, then $q_\kappa(\mathbf{A}) = 1$.

4. if $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then

$$q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A}) q_\tau(\mathbf{A}), \qquad (7)$$

   in particular if $\tau = (p, p, \dots, p)$, then

$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A}). \qquad (8)$$

5. Finally, for $\mathbf{B} \in \mathfrak{F}^{m \times m}$ such that $\mathbf{C} = \mathbf{B}^* \mathbf{B} \in \mathfrak{S}_m^\beta$,

$$q_\kappa(\mathbf{B}\mathbf{A}\mathbf{B}^*) = q_\kappa(\mathbf{C}) q_\kappa(\mathbf{A}) \qquad (9)$$

   and

$$q_\kappa(\mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{*-1}) = (q_\kappa(\mathbf{C}))^{-1} q_\kappa(\mathbf{A}). \qquad (10)$$

Remark 2.1. Let $\mathcal{P}(\mathfrak{S}_m^\beta)$ denote the algebra of all polynomial functions on $\mathfrak{S}_m^\beta$, and $\mathcal{P}_k(\mathfrak{S}_m^\beta)$ the subspace of homogeneous polynomials of degree k and let $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m^\beta)$ such that

$$\mathcal{P}_k(\mathfrak{S}_m^\beta) = \sum_\kappa \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m^\beta).$$

Note that q_κ is a homogeneous polynomial of degree k, moreover $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$, see Gross and Richards (1987).

In (2), $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ, defined as

$$[a]_\kappa^\beta = \prod_{i=1}^{m}(a-(i-1)\beta/2)_{k_i} = \frac{\Gamma_m^\beta[a,\kappa]}{\Gamma_m^\beta[a]},$$

where $\mathrm{Re}(a) > (m-1)\beta/2-k_m$ and $(a)_i = a(a+1)\cdots(a+i-1)$, is the standard Pochhammer symbol.

A variant of the generalised gamma function of weight κ is obtained from Khatri (1966) and is defined as

$$\begin{aligned}\Gamma_m^\beta[a,-\kappa] &= \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{A}^{-1})(d\mathbf{A}) \qquad (11)\\ &= \pi^{m(m-1)\beta/4}\prod_{i=1}^{m}\Gamma[a-k_i-(i-1)\beta/2]\\ &= \frac{(-1)^k\Gamma_m^\beta[a]}{[-a+(m-1)\beta/2+1]_\kappa^\beta}, \qquad (12)\end{aligned}$$

where $\mathrm{Re}(a) > (m-1)\beta/2+k_1$.

Now, we show three Jacobians in terms of the β parameter, which are proposed as extensions of real, complex or quaternion cases, see Díaz-García and Gutiérrez-Jáimez (2011).

Lemma 2.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}\in\mathfrak{P}_m^\beta$ *matrices of functionally independent variables, and let* $\mathbf{Y}=\mathbf{AXA}^*+\mathbf{C}$, *where* $\mathbf{A}\in\mathcal{L}_{m,m}^\beta$ *and* $\mathbf{C}\in\mathfrak{P}_m^\beta$ *are matrices of constants. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{\beta(m-1)/2+1}(d\mathbf{X}). \qquad (13)$$

Lemma 2.2. *Let* $\mathbf{S}\in\mathfrak{P}_m^\beta$. *Then ignoring the sign, if* $\mathbf{Y}=\mathbf{S}^{-1}$

$$(d\mathbf{Y}) = |\mathbf{S}|^{-\beta(m-1)-2}(d\mathbf{S}). \qquad (14)$$

We end this section with a some general results, which are useful in a variety of situations, which enable us to transform the density function of a matrix $\mathbf{X}\in\mathfrak{P}_m^\beta$ to the density function of its eigenvalues, see Díaz-García and Gutiérrez-Jáimez (2009).

Lemma 2.3. *Let* $\mathbf{X}\in\mathfrak{P}_m^\beta$ *be a random matrix with density function* $f_{\mathbf{X}}(\mathbf{X})$. *Then the joint density function of the eigenvalues* $\lambda_1,\dots,\lambda_m$ *of* $\mathbf{X}$ *is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^\beta\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} f(\mathbf{HLH}^*)(d\mathbf{H}) \qquad (15)$$

where $\mathbf{L}=\mathrm{diag}(\lambda_1,\dots,\lambda_m)$, $\lambda_1>\cdots>\lambda_m>0$, $(d\mathbf{H})$ *is the normalised Haar measure and*

$$\varrho = \begin{cases} 0, & \beta=1;\\ -m, & \beta=2;\\ -2m, & \beta=4;\\ -4m, & \beta=8.\end{cases}$$

3 Riesz distributions

Alternatively to find the Riesz distribution from the Riesz measure, Hassairi and Lajmi (2001), in this section we establish the two versions of the Riesz distribution via the next result. For this purpose, we utilise the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$. That

is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consist of all matrices $\mathbf{X} \in (\mathfrak{F}^{\mathfrak{C}})^{m\times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^{\beta}$. We refer to $\mathbf{X} = \mathrm{Re}(\mathbf{Z})$ and $\mathbf{Y} = \mathrm{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi}_m^{\beta} = \mathfrak{P}_m^{\beta} + i\mathfrak{S}_m^{\beta}$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ such that $\mathrm{Re}(\mathbf{Z}) \in \mathfrak{P}_m^{\beta}$, see (Gross and Richards, 1987, p. 801).

Lemma 3.1. *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^{\beta}$ *and* $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$. *Then*

1. *for* $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$,

$$\int_{\mathbf{A}\in\mathfrak{P}_m^{\beta}} \mathrm{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{A})(d\mathbf{A})$$
$$= \Gamma_m^{\beta}[a,\kappa]|\mathbf{\Sigma}|^{a} q_{\kappa}(\mathbf{\Sigma}) \tag{16}$$

2. *for* $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$,

$$\int_{\mathbf{A}\in\mathfrak{P}_m^{\beta}} \mathrm{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{A}^{-1})(d\mathbf{A})$$
$$= \frac{\Gamma_m^{\beta}[a,-\kappa]|\mathbf{\Sigma}|^{a}}{q_{\kappa}(\mathbf{\Sigma})} \tag{17}$$

Proof. Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^{\beta}$ and in integrals (16) and (17) make the change of variable $\mathbf{A} = \mathbf{\Sigma}^{1/2}\mathbf{W}\mathbf{\Sigma}^{1/2}$, where $\mathbf{\Sigma}^{1/2} \in \mathbf{\Phi}_m^{\beta}$ denotes the square root of $\mathbf{\Sigma}$, such that $(\mathbf{\Sigma}^{1/2})^2 = \mathbf{\Sigma}$. By Lemma 2.1, $(d\mathbf{A}) = |\mathbf{\Sigma}|^{\beta(m-1)/2+1}(d\mathbf{W})$, then integrals becomes

1. $$\int_{\mathbf{W}\in\mathfrak{P}_m^{\beta}} \mathrm{etr}\{-\mathbf{W}\}|\mathbf{W}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{\Sigma}^{1/2}\mathbf{W}\mathbf{\Sigma}^{1/2})|\mathbf{\Sigma}|^{a}(d\mathbf{W}).$$

2. and $$\int_{\mathbf{W}\in\mathfrak{P}_m^{\beta}} \mathrm{etr}\{-\mathbf{W}\}|\mathbf{W}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{\Sigma}^{-1/2}\mathbf{W}^{-1}\mathbf{\Sigma}^{-1/2})|\mathbf{\Sigma}|^{a}(d\mathbf{W}).$$

The desired results now is follows from (1) and (9), and (11) and (10), respectively. □

Hence, as consequence of Lemma 3.1, now we can propose the two definitions of Riesz distribution.

Definition 3.1. Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^{\beta}$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.

1. Then it said that $\mathbf{X}$ has a Riesz distribution of type I if its density function is

$$\frac{1}{\Gamma_m^{\beta}[a,\kappa]|\mathbf{\Sigma}|^{a} q_{\kappa}(\mathbf{\Sigma})} \mathrm{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{X})(d\mathbf{X}) \tag{18}$$

   for $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$.

2. Then it said that $\mathbf{X}$ has a Riesz distribution of type II if its density function is

$$\frac{q_{\kappa}(\mathbf{\Sigma})}{\Gamma_m^{\beta}[a,-\kappa]|\mathbf{\Sigma}|^{a}} \mathrm{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{X}^{-1})(d\mathbf{X}) \tag{19}$$

   for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$.

Note that, the matrix multivariate gamma distribution is a particular example of the Riesz distribution. Furthermore, if $\kappa = (0, 0, \dots, 0)$ in two densities in Definition 3.1 the matrix multivariate gamma distribution is obtained. In addition observe that, in this last case if $\mathbf{X}$ is defined as $\mathbf{Y} = 2\mathbf{X}$ the Wishart distribution is gotten.

In next result is established their characteristic functions of Riesz distributions.

Theorem 3.1. *Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.*

1. *Then if $\mathbf{X}$ has a Riesz distribution of type I its characteristic function is*

$$|\mathbf{I}_m - i\mathbf{\Sigma}\mathbf{T}|^{-a} q_\kappa\left(\left(\mathbf{I}_m - i\mathbf{\Sigma}^{1/2}\mathbf{T}\mathbf{\Sigma}^{1/2}\right)^{-1}\right) \tag{20}$$

for $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$.

2. *Then if $\mathbf{X}$ has a Riesz distribution of type II its characteristic function is*

$$\frac{|\mathbf{I}_m - i\mathbf{\Sigma}\mathbf{T}|^{-a}}{q_\kappa\left(\left(\mathbf{I}_m - i\mathbf{\Sigma}^{1/2}\mathbf{T}\mathbf{\Sigma}^{1/2}\right)^{-1}\right)} = |\mathbf{I}_m - i\mathbf{\Sigma}\mathbf{T}|^{-a} q_\kappa\left(\left(\mathbf{I}_m - i\mathbf{\Sigma}^{1/2}\mathbf{T}\mathbf{\Sigma}^{1/2}\right)\right) \tag{21}$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$.

Proof. 1. The characteristic function is

$$\frac{1}{\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})}\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-(\mathbf{\Sigma}^{-1} - i\mathbf{T})\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{X})(d\mathbf{X}).$$

From (16) the characteristic function is

$$\frac{|\mathbf{\Sigma}^{-1} - i\mathbf{T}|^{-a} q_\kappa((\mathbf{\Sigma}^{-1} - i\mathbf{T})^{-1})}{|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} = \frac{|\mathbf{I}_m - i\mathbf{T}\mathbf{\Sigma}|^{-a} q_\kappa((\mathbf{\Sigma}^{-1} - i\mathbf{T})^{-1})}{q_\kappa(\mathbf{\Sigma})}$$

The final expression is obtained from (10). The results stated in 2 is obtained in analogously. □

Finally, it is found the joint distributions of the eigenvalues for random matrices Riesz type I and II.

Theorem 3.2. *Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$.*

1. *Let $\lambda_1, \dots, \lambda_m$, $\lambda_1 > \cdots > \lambda_m > 0$ be the eigenvalues of $\mathbf{X}$. Then if $\mathbf{X}$ has a Riesz distribution of type I, the joint density of $\lambda_1, \dots, \lambda_m$ is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})}\prod_{i<j}^m(\lambda_i - \lambda_j)^\beta$$

$$\times \prod_{i=1}^m \lambda_i^{a+k_i-(m-1)\beta/2-1} {}_0F_0^{(m),\beta}(-\mathbf{\Sigma}^{-1}, \mathbf{L}). \tag{22}$$

where $\mathbf{L} = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$ *and* $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$.

2. *Let $\delta_1, \dots, \delta_m$, $\delta_1 > \cdots > \delta_m > 0$ be the eigenvalues of $\mathbf{X}$. Then if $\mathbf{X}$ has a Riesz distribution of type II, the joint density of their eigenvalues is*

$$\frac{\pi^{m^2\beta/2+\varrho} q_\kappa(\mathbf{\Sigma})}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a}\prod_{i<j}^m(\delta_i - \delta_j)^\beta$$

$$\times \prod_{i=1}^{m} \delta_i^{a-k_i-(m-1)\beta/2-1} {}_0F_0^{(m),\beta}(-\mathbf{\Sigma}^{-1}, \mathbf{D}). \qquad (23)$$

where $\mathbf{D} = \operatorname{diag}(\delta_1, \dots, \delta_m)$ *and* $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$.

And where

$$\begin{aligned}\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{\mathbf{AHBH}^*\}(d\mathbf{H}) &= \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} {}_0F_0^{\beta}(\mathbf{AHBH}^*)(d\mathbf{H}) \\ &= {}_0F_0^{(m),\beta}(\mathbf{A}, \mathbf{B}),\end{aligned}$$

see Díaz-García and Gutiérrez-Jáimez (2009, Eq. 4.9, p. 18) and James (1964, Eq. 60, p.483).

Proof. 1. From Lemma 2.3

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^\beta$$

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{HLH}^*\}|\mathbf{HLH}^*|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{HLH}^*)(d\mathbf{H}).$$

Therefore, by (4) and (9),

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^\beta \prod_{i=1}^{m} \lambda_i^{a+k_i-(m-1)\beta/2-1}$$

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{HLH}^*\}(d\mathbf{H}).$$

The desired result is obtained from Díaz-García and Gutiérrez-Jáimez (2009, Eq. 4.9, p. 18), see also James (1964, Eq. 60, p.483), observing that

$$\begin{aligned}\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{-\mathbf{\Sigma}^{-1}\mathbf{HLH}^*\}(d\mathbf{H}) &= \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} {}_0F_0^{\beta}(-\mathbf{\Sigma}^{-1}\mathbf{HLH}^*)(d\mathbf{H}) \\ &= {}_0F_0^{(m),\beta}(-\mathbf{\Sigma}^{-1}, \mathbf{L}).\end{aligned}$$

2. is proved similarly. □

Again, observe that when $\kappa = (0, \dots, 0)$, the corresponding joint densities of the eigenvalues of the matrix multivariate gamma and Wishart distributions are obtained as particular cases of (22) and (23), indistinctly.

4 Conclusions

Finally, note that the real dimension of real normed division algebras can be expressed as powers of 2, $\beta = 2^n$ for $n = 0, 1, 2, 3$. On the other hand, as it can be check in Kabe (1984), the results obtained in this work can be extended to the hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra), but are Jordan algebras, and all their isomorphic algebras. Note, also, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras.

Thus, the results for hypercomplex algebras are obtained by simply replacing β with 2β in our results. Alternatively, following Kabe (1984), we can conclude that, our results are true for '2^n-ions', $n = 0, 1, 2, 3, 4, 5$, emphasising that only for $n = 0, 1, 2, 3$ the corresponding algebras are real normed division algebras.

Here we prefer our notation used unlike the proposal in Hassairi and Lajmi (2001), because we think that is clearer to a statistician. Note, however, that any of the results obtained can be written easily as proposed by Hassairi and Lajmi (2001) in the context of Euclidean simple Jordan algebras, simply applying (8). For example in this case (20) can be written equivalently as:

$$q_{\kappa+a}\left(\left(\mathbf{I}_m - i\mathbf{\Sigma}^{1/2}\mathbf{T}\mathbf{\Sigma}^{1/2}\right)^{-1}\right).$$

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